Directions of directional, ordered directional and strengthened ordered directional increasingness of linear and ordered linear fusion operators

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***Abstract*—In this work we discuss the forms of monotonicity that have been recently introduced to relax the monotonicity condition in the definition of aggregation functions. We focus on directional, ordered directional and strengthened ordered directional monotonicity, study their main properties and provide some results about their links and relations among them. We also present two families of functions, the so-called linear fusion functions and ordered linear fusion functions and we study the set of directions for which these types of functions are directionally, ordered directionally and strengthened ordered directionally increasing. In particular, OWA operators are an example of ordered linear fusion functions.**

***Index Terms*—Aggregation function, directional monotonicity, generalizations of monotonicity, OWA operator**

I. Introduction

A function $A : [0,1]^n \to [0,1]$ such that $A(\mathbf{0}) = 0$, $A(\mathbf{1}) = 1$ and it is component-wise increasing is said to be an aggregation function [2], [12]. The aim of aggregation functions is to find a representative number for n inputs, or, in other words, to fuse or aggregate information. The need of fusing information is common to nearly every process that utilizes data. In fact, aggregation functions have been greatly studied theoretically [1], [6], [10] and successfully used in various applications [9], [11], [17], [21].

Recently, some authors have proposed that the axiom of monotonicity of aggregation functions can be sometimes too restrictive and can leave out of the theoretical framework functions that are valid to fuse data. As an example, in [22] fuzzy implication operators [8], the mode operator, Lehmer means [3], etc. are mentioned.

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On that account, several proposals for the relaxation of the monotonicity condition of aggregation functions have emerged. The first notion that arised is the so-called weak monotonicity in [22]. Weak monotonicity focuses on the increasingness of the value of a function whenever all the inputs increase by the same amount. This can be understood as monotonicity along the ray given by $\vec{1} = (1, \ldots, 1)$. Thus, considering any non-zero vector $\vec{r} \in \mathbb{R}^n$ has led to the definition of directional monotonicity [7]. The concept of directional monotonicity, in turn, has led to the definition of pre-aggregation functions [15], which are functions satisfying the same conditions as aggregation functions but that are directionally increasing for some direction $\vec{r}$ rather than with respect to every argument. Pre-aggregation functions have been applied to the problem of classification, specifically in the fuzzy ruled based classification systems setting [13], [14].

Both weak and directional monotonicity consider the same ray of increasingness for all the points in the domain of a function f. Recently, two additional forms of monotonicity have been introduced for which the direction of increasingness varies depending on the specific point to aggregate. These forms of monotonicity are called ordered directional (OD) monotonicity and strengthened ordered directional (SOD) monotonicity and were introduced in [5] and [20], respectively. In both forms, the relative size of each input affects the direction along which an ordered directionally, or strengthened ordered directionally, increasing function increases. OD monotone functions have been applied in edge detection algorithms for computer vision [16], [18].

In this work, we discuss each of these weaker forms of monotonicity. We study a collection of the properties that they satisfy and we expose the relations that exist between each form of monotonicity, as well as some methods to construct functions of this type. Subsequently, we present two families of functions that generalize OWA operators: linear fusion functions and ordered linear fusion functions. We discuss some of their properties and show some instances of such functions. Moreover, we study and characterize the set of directions for which a given linear fusion function f is increasing in the sense of directional, ordered directional and strengthened ordered directional monotonicity. We also characterize the set of direction for which an ordered linear fusion function is increasing for the two-dimensional case. Finally, we show that OWA operators [23], that are of great utility in decision making processes, are a particular case of ordered linear fusion functions.

This paper is organized as follows. In Section II we recall the preliminary notions that are needed throughout the rest of the paper. In Section III we discuss weak, directional, ordered directional and strengthened ordered directional monotonicity, defining each notion and providing some examples. In Section IV, we study the properties of each form of monotonicity and we expose some results about their relation with each other. In Section V we present the classes of linear fusion functions and ordered linear fusion functions, that generalize OWA operators, and we show some examples of such functions. We finish the work in Section VI with some concluding remarks.

II. Preliminaries

Let us start recalling the definition of aggregation function.

Definition 1: We say that a function $A\colon [0,1]^n \to [0,1]$ is an aggregation function if

1) $A(0,\dots,0)=0$ and $A(1,\dots,1)=1$;
2) A is increasing, i.e., if $(x_1,\dots,x_n),(y_1,\dots,y_n) \in [0,1]^n$ such that $x_i \leq y_i$ for all $i \in \{1,\dots,n\}$, then $A(x_1,\dots,x_n) \leq A(y_1,\dots,y_n)$.

In this work, we deal with points $\mathbf{x} \in [0,1]^n$, directions $\vec{r} \in \mathbb{R}^n$ and we use the component-wise partial order $\leq_L$ on $[0,1]^n$:

$$\mathbf{x} \leq_L \mathbf{y} \iff x_i \leq y_i \text{ for all } i \in \{1,\dots,n\},$$

where $\mathbf{x},\mathbf{y} \in [0,1]^n$.

We also apply permutations to the components of a given n-tuples $\mathbf{x} \in [0,1]^n$ and $\vec{r} \in \mathbb{R}^n$. If we denote by S_n the set of all permutations of the set $\{1,\dots,n\}$, and consider $\sigma \in S_n$, we set the following notation:

$$\mathbf{x}_\sigma = (x_{\sigma(1)},\dots,x_{\sigma(n)}) \in [0,1]^n$$
$$\vec{r}_\sigma = (r_{\sigma(1)},\dots,r_{\sigma(n)}) \in \mathbb{R}^n.$$

Clearly, given $\mathbf{x},\mathbf{y} \in [0,1]^n$ and $\sigma \in S_n$, the following assertions hold:

- $(\mathbf{x}+\mathbf{y})_\sigma = \mathbf{x}_\sigma + \mathbf{y}_\sigma$;
- $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}_\sigma \cdot \mathbf{y}_\sigma$,

where $\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{n} x_i y_i$.

For the subsequent developments, it is useful to set a notation for the sets whose elements are ordered in an decreasing (or increasing) manner. If $H \subset \mathbb{R}^n$, we denote

$$H_{(\geq)} = \{(h_1,\dots,h_n) \in H \mid h_1 \geq \cdots \geq h_n\}.$$

Analogously, one can define the subsets $H_{(\leq)}$, $H_{(>)}$, $H_{(<)}$ and $H_{(=)}$.

III. Four different forms of monotonicity

The first notion of monotonocity that we discuss in this work was given in [22] with the aim of creating a framework for functions that are adequate for aggregating data but do not satisfy the monotonicity condition of aggregation functions. This type of monotonicity is called weak monotonicity.

Definition 2: We say that a function $f : [0,1]^n \to [0,1]$ is weakly increasing (resp. weakly decreasing) if for all $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $0 \leq x_i + c \leq 1$ for all $i \in \{1,\dots,n\}$, it holds that $f(x_1+c,\dots,x_n+c) \geq f(x_1,\dots,x_n)$ (resp. $f(x_1+c,\dots,x_n+c) \leq f(x_1,\dots,x_n)$).

The second form of monotonicity that we discuss is directional monotonicity [7].

Definition 3: Let $f\colon [0,1]^n \to [0,1]$ and $\vec{0} \neq \vec{r} \in \mathbb{R}^n$, we say that F is $\vec{r}$-increasing (resp. $\vec{r}$-decreasing), if for all $c > 0$ and $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x}+c\vec{r} \in [0,1]^n$, it holds that $F(\mathbf{x}+c\vec{r}) \geq F(\mathbf{x})$ (resp. $F(\mathbf{x}+c\vec{r}) \leq F(\mathbf{x})$).

If we take the vector $\vec{r} = (1,\dots,1)$, we recover weak monotonicity. Therefore, weak monotonicity is a particular case of directional monotonicity.

Directional monotonicity can be understood as monotonicity along a certain ray in the domain, given by the vector $\vec{r}$. This direction is the same for all the points in the domain. In the remaining two forms of monotonicity that we discuss in this work the direction of increasingness varies from one point to another.

Example 1:

Let $0 < \lambda < 1$ and let $L_\lambda : [0,1]^2 \to [0,1]$ be the function given by

$$L_\lambda(x,y) = \frac{\lambda x^2 + (1-\lambda)y^2}{\lambda x + (1-\lambda)y},$$

with the convention $\frac{0}{0} = 0$. This function is called weighted Lehmer mean and in [7] it is shown that the only direction along which it increases is $\vec{r} = (1-\lambda,\lambda)$, up to positive multiplicative constant.

The third notion that we discuss is ordered directional (OD) monotonicity [5].

Definition 4: Given $\vec{0} \neq \vec{r} \in \mathbb{R}^n$, a function $f : [0,1]^n \to [0,1]$ is said to be ordered directionally, OD, $\vec{r}$-increasing (resp. OD $\vec{r}$-decreasing), if for all $c > 0$, $\sigma \in S_n$ and $\mathbf{x} \in [0,1]^n$, it holds that if $\mathbf{x}_\sigma,\ \mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)}$, then $f(\mathbf{x}+c\vec{r}_{\sigma^{-1}}) \geq f(\mathbf{x})$ (resp. $f(\mathbf{x}+c\vec{r}_{\sigma^{-1}}) \leq f(\mathbf{x})$).

Specifically, the direction of increasingness of an ordered directionally monotone function depends on the relative size of each component of the input.

If we were to check if a specific function is OD $\vec{r}$-increasing for some $\vec{r} \in \mathbb{R}^n$, we need to check condition for the points

$\mathbf{x} \in [0,1]^n$ that satisfy $\mathbf{x}_\sigma,\ \mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)}$. The fourth form of monotonicity results from not requiring the condition $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)}$, but $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$ instead. It is called strengthened ordered directional (SOD) monotonicity and it was introduced in [20].

Definition 5: Given $\vec{0} \neq \vec{r} \in \mathbb{R}^n$, a function $f : [0,1]^n \to [0,1]$ is said to be strengthened ordered directionally, SOD, $\vec{r}$-increasing (resp. SOD $\vec{r}$-decreasing), if for all $c > 0$, $\sigma \in S_n$ and $\mathbf{x} \in [0,1]^n$, it holds that if $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$, then $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\mathbf{x})$ (resp. $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \leq F(\mathbf{x})$).

A function f that is at the same time $\vec{r}$-increasing and $\vec{r}$-decreasing is said to be $\vec{r}$-constant. Similarly, a function can be OD $\vec{r}$-constant and SOD $\vec{r}$-constant.

Example 2: Let $p > 0$ and $\vec{r} = (t, \ldots, t, s) \in \mathbb{R}^n$ with $s \leq t$. The function $F\colon [0,1]^n \to [0,1]$ given by

$$F(\mathbf{x}) = \frac{1}{n-1} \sum_{j=2}^{n} |x_1 - x_j|^p \,,$$

for $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$, is SOD $\vec{r}$-increasing.

Indeed, let $\mathbf{x} \in [0,1]^n$, $\sigma \in S_n$ and $c > 0$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$. First, if $\sigma(n) = 1$, $x_1 \leq x_j$ for all $j > 1$. And since $s \leq t$, the result follows from

$$|x_1 - x_j + cs - ct| \geq |x_1 - x_j|.$$

Now, if $1 \neq i = \sigma(n)$, it holds that $x_1 \geq x_i$ for all $i > 1$ and since $t \geq s$, we have

$$\begin{aligned} &F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) = \\ &= \frac{1}{n-1} \left(\sum_{\substack{j=2 \\ j \neq i}}^{n} |x_1 - x_j|^p + |(x_1 - x_i) + c(t-s)|^p \right) \\ &\qquad\qquad \geq \frac{1}{n-1} \sum_{j=2}^{n} |x_1 - x_j|^p = F(\mathbf{x})\,. \end{aligned}$$

IV. Facts and relation between the different forms of monotonicity

Although we have distinguished four different forms of monotonicity, we present the developments for the last three, since weak monotonicity can be understood as a particular case of directional monotonicity.

A relation between the discussed forms of monotonicity that comes readily is a consequence of the change of the definition of OD monotonicity to define SOD monotonicity. While a function f is required to fulfill the inequality $f(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \geq f(\mathbf{x})$ for some $\mathbf{x} \in hcn$ such that

$$\mathbf{x}_\sigma, \mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)} \tag{1}$$

in order to be considered OD increasing, f is required to fulfill the same inequality for points $\mathbf{x} \in hc$ that satisfy the relation in (1) and also points such that $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$ (it may happen that $\mathbf{x}_\sigma + c\vec{r} \notin [0,1]^n_{(\geq)}$) in order to be considered SOD increasing. Consequently, if a function f is SOD, then it is OD increasing. The converse statement is not true. Equivalently, the set of vectors $\vec{r}$ for which a function f is SOD $\vec{r}$-increasing (resp. SOD $\vec{r}$-decreasing and SOD $\vec{r}$-constant) is contained in the set of vectors for which f is OD $\vec{r}$-increasing (resp. OD $\vec{r}$-decreasing and OD $\vec{r}$-constant).

However, for some directions $\vec{r}$, the notions of OD monotonicity and SOD monotonicity are equivalent.

Proposition 1: Let $\vec{r} \in \mathbb{R}^n_{(\geq)}$. Then, a function f is OD $\vec{r}$-increasing (resp. OD $\vec{r}$-decreasing) if and only if f is SOD $\vec{r}$-increasing (resp. SOD $\vec{r}$-decreasing).

Clearly, Proposition 1 holds also true for the OD and SOD $\vec{r}$-constant case.

Another peculiarity of each form of monotonicity is the set of points for which there is no need to check the inequality in each definition. These points are the ones that do not satisfy the conditions that must be fulfilled in order to satisfy the monotonicity condition. When we are interested in directional monotonicity, this set of points $\mathbf{x} \in [0,1]^n$ are characterized by the relation $\mathbf{x} + c\vec{r} \notin [0,1]^n$ for every $c > 0$. On the contrary, for a OD $\vec{r}$-increasing function f, a point $\mathbf{x} \in [0,1]^n$ is of this type if $\mathbf{x}_\sigma + c\vec{r} \notin [0,1]^n_{(\geq)}$ for $\sigma \in S_n$ with $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$. Finally, if f is SOD $\vec{r}$-increasing, then a point $\mathbf{x} \in [0,1]^n$ is of this type if $\mathbf{x}_\sigma + c\vec{r} \notin [0,1]^n$ for $\sigma \in S_n$ with $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$.

For an explicit description of this type of points for each possible direction $\vec{r} \in \mathbb{R}^n$, see [19].

It is worth to mention that it is equivalent to increase a direction $\vec{r}$ and the direction resulting from a positive scalar multiplication. This holds for the three forms of monotonicity as the following result asserts.

Proposition 2: Let $f\colon [0,1]^n \to [0,1]$ and $k > 0$. Then, the following items hold

(a) f is $\vec{r}$-increasing if and only if f is $(k\vec{r})$-increasing;
(b) f is OD $\vec{r}$-increasing if and only if f is OD $(k\vec{r})$-increasing;
(c) f is SOD $\vec{r}$-increasing if and only if f is SOD $(k\vec{r})$-increasing.

A consequence of Proposition 2 is that we could choose a representative direction $\vec{r} \in \mathbb{R}^n$ to refer to all the directions $\alpha\vec{r}$ for $\alpha > 0$. Consequently, from this point on, we only consider directions $\vec{r} \in \mathbb{R}^n$ such that

$$\|\vec{r}\|_2 = \sqrt{r_1^2 + \ldots + r_n^2} = 1,$$

unless otherwise stated. Note that this is possible due to the fact that for an arbitrary $\vec{r} \in \mathbb{R}^n$, it holds that $\|\frac{1}{\|\vec{r}\|_2}\vec{r}\|_2 = 1$ and since $\frac{1}{\|\vec{r}\|_2} > 0$, we can apply Proposition 2.

In the following result, we show a property that holds for functions that are $\vec{r}$-increasing and for functions that are OD $\vec{r}$-increasing, but not for functions that are SOD $\vec{r}$-increasing.

Proposition 3: Let $f\colon [0,1]^n \to [0,1]$ and $\vec{r} \in \mathbb{R}^n$. The following items hold

(a) f is $\vec{r}$-increasing if and only if f is $(-\vec{r})$-decreasing;
(b) f is OD $\vec{r}$-increasing if and only if f is OD $(-\vec{r})$-decreasing.

Proof:

(a):
Let f be $\vec{r}$-increasing and let $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $\mathbf{x} - c\vec{r} \in [0,1]^n$. Now, since f is $\vec{r}$-increasing, it holds that

$$F(\mathbf{x} + c(-\vec{r})) \leq F(\mathbf{x} + c(-\vec{r}) + c\vec{r}) = F(\mathbf{x}),$$

and therefore f is $(-\vec{r})$-decreasing. The reverse implication is analogous.

(b):
Let f be an OD $\vec{r}$-increasing function and let $\mathbf{x} \in [0,1]^n$, $c > 0$ and $\sigma \in S_n$ such that $\mathbf{x}_\sigma$ and $\mathbf{x}_\sigma + c(-\vec{r}) \in [0,1]^n_{(\geq)}$. Note that since $\mathbf{x}_\sigma + c(-\vec{r}) + c\vec{r} = \mathbf{x}_\sigma$, it holds that $\mathbf{x}_\sigma + c(-\vec{r}) + c\vec{r} \in [0,1]^n_{(\geq)}$. Now, the fact that f is OD $\vec{r}$-increasing implies that f is OD $\vec{r}$-decreasing. The reverse implication is analogous. ■

However, as it is shown in [20], this is not so for a function f that is SOD $\vec{r}$-increasing.

There exist some properties that are shared by the three different forms of monotonicity. One of the most relevant ones is that if a function f is increasing along two directions $\vec{r}$ and $\vec{s}$ (either directionally, ordered directionally or strengthened ordered directionally), then it increases (in the same sense) along any direction formed by a positive convex combination of $\vec{r}$ and $\vec{s}$. We illustrate this fact in the following three results, which were presented in [7], [5] and [20], respectively.

Theorem 1 ([7]): Let $\vec{r}, \vec{s} \in \mathbb{R}^n$ and $a, b > 0$. Let $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that if $\mathbf{x}$ and $\mathbf{x} + c(a\vec{r} + b\vec{s}) \in [0,1]^n$, then either $\mathbf{x} + ca\vec{r}$ or $\mathbf{x} + cb\vec{s} \in [0,1]^n$. Then, a function $f : [0,1]^n \to [0,1]$ that is $\vec{r}$-increasing and $\vec{s}$-increasing simultaneously is also $(a\vec{r} + b\vec{s})$-increasing.

Theorem 2 ([5]): Let $\vec{r}, \vec{s} \in \mathbb{R}^n$ and $a, b > 0$. Let $\mathbf{x} \in [0,1]^n$, $c > 0$, $\sigma \in S_n$ such that if $\mathbf{x}_\sigma$ and $\mathbf{x}_\sigma + c(a\vec{r} + b\vec{s}) \in [0,1]^n_{(\geq)}$, then either $\mathbf{x} + ca\vec{r}$ or $\mathbf{x} + cb\vec{s} \in [0,1]^n_{(\geq)}$. Then, a function $f : [0,1]^n \to [0,1]$ that is OD $\vec{r}$-increasing and OD $\vec{s}$-increasing simultaneously is also OD $(a\vec{r} + b\vec{s})$-increasing.

Theorem 3 ([20]): Let $\vec{r}, \vec{s} \in \mathbb{R}^n$ and $a, b > 0$. Let $\mathbf{x} \in [0,1]^n$, $c > 0$, $\sigma \in S_n$ such that if $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and $\mathbf{x}_\sigma + c(a\vec{r} + b\vec{s}) \in [0,1]^n$, then either $\mathbf{x} + ca\vec{r}$ or $\mathbf{x} + cb\vec{s} \in [0,1]^n$. Then, a function $f : [0,1]^n \to [0,1]$ that is SOD $\vec{r}$-increasing and SOD $\vec{s}$-increasing simultaneously is also SOD $(a\vec{r} + b\vec{s})$-increasing.

Let us now present two additional results that show how, once we have a function that satisfies one of the discussed forms of monotonicity, we can construct new functions that satisfy the same type of monotonicity.

Proposition 4: Let $f \colon [0,1]^n \to [0,1]$ be a $\vec{r}$-increasing function and let $\varphi \colon [0,1] \to [0,1]$ be an increasing (resp. decreasing) function. Then, the composition $\varphi \circ f \colon [0,1]^n \to [0,1]$ is $\vec{r}$-increasing (resp. $\vec{r}$-decreasing).

Theorem 4: Let $\vec{r} \in \mathbb{R}^n$ and $f_i \colon [0,1]^n \to [0,1]$, $1 \leq i \leq m$, be m $\vec{r}$-increasing functions (resp. $\vec{r}$-decreasing). Let $A : [0,1]^n \to [0,1]$ be an aggregation function. Then the function $A\left(f_1, \ldots, f_m\right) : [0,1]^n \to [0,1]$, given by $A\left(f_1, \ldots, f_m\right)(\mathbf{x}) = A(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, is $\vec{r}$-increasing (resp. $\vec{r}$-decreasing).

For example, the arithmetic mean of m different directionally (resp. OD, SOD) monotone functions is another directionally (resp. OD, SOD) monotone function.

Note that both Proposition 4 and Theorem 4 can be equivalently stated and holds for the cases of OD and SOD $\vec{r}$-increasingness.

Let us end this section about some facts and relations between the three forms of monotonicity by presenting a theorem that relates every notion with each other, including standard monotonicity. It characterizes standard monotonicity by means of directional, OD and SOD increasingness along certain directions.

Theorem 5 ([20]): Let $f \colon [0,1]^n \to [0,1]$ and $(\vec{e}_1, \ldots, \vec{e}_n)$ be the set of vectors such that $\vec{e}_i$ is given by 1 in the i-th component and 0 in the remaining components. Then, the following are equivalent:

(a) f is increasing;
(b) f is $\vec{e}_i$-increasing for all $i \in \{1, \ldots, n\}$;
(c) f is OD $\vec{e}_i$-increasing for all $i \in \{1, \ldots, n\}$;
(d) f is SOD $\vec{e}_i$-increasing for all $i \in \{1, \ldots, n\}$.

V. The set of directions for which a function is monotone

In this section we study the sets of vectors $\vec{r} \in \mathbb{R}^n$ for which a given function f is $\vec{r}$-increasing, OD $\vec{r}$-increasing and SOD $\vec{r}$-increasing. The notation we use to refer to these sets for a function $f : [0,1]^n \to [0,1]$ is the following:

$$\begin{aligned}
\mathcal{C}(F) &= \{\, \vec{r} \in \mathbb{R}^n \mid F \text{ is } \vec{r}\text{-constant} \,\}, \\
\mathcal{C}_{OD}(F) &= \{\, \vec{r} \in \mathbb{R}^n \mid F \text{ is OD } \vec{r}\text{-constant} \,\}, \\
\mathcal{C}_{SOD}(F) &= \{\, \vec{r} \in \mathbb{R}^n \mid F \text{ is SOD } \vec{r}\text{-constant} \,\}, \\
\mathcal{D}^{\uparrow}(F) &= \{\, \vec{r} \in \mathbb{R}^n \mid F \text{ is } \vec{r}\text{-increasing} \,\}, \\
\mathcal{D}^{\uparrow}_{OD}(F) &= \{\, \vec{r} \in \mathbb{R}^n \mid F \text{ is OD } \vec{r}\text{-increasing} \,\}, \\
\mathcal{D}^{\uparrow}_{SOD}(F) &= \{\, \vec{r} \in \mathbb{R}^n \mid F \text{ is SOD } \vec{r}\text{-increasing} \,\}.
\end{aligned}$$

In particular, we study the aforementioned sets for the class of functions of the following definition.

Definition 6: Let $\mu \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$, and let $L[\mu, \vec{v}] : [0,1]^n \to [0,1]$ be a function given by

$$L[\mu, \vec{v}]\,(\mathbf{x}) = \mu + \mathbf{x} \cdot \vec{v} = \mu + \sum_{i=1}^{n} x_i v_i,$$

such that $\mu + \mathbf{x} \cdot \vec{v} \in [0,1]$ for all $\mathbf{x} \in [0,1]^n$. We call the function $L[\mu, \vec{v}]$ a linear fusion function.

We next present a characterization of this class of functions in terms of the constant μ and the vector $\vec{v}$.

Proposition 5: $L[\mu, \vec{v}]$ is a linear fusion function if and only if

$$0 \leq \mu + \sum_{i \in H} v_i \leq 1$$

for all $H \subset \{1, \ldots, n\}$.

Corollary 1: If $L[\mu, \vec{v}]$ is a linear fusion function, the function $L[1 - \mu, -\vec{v}]$ is also a linear fusion function.

Proof: It is easy to check that for every $H \subset \{1, \dots, n\}$

$$0 \leq \mu + \sum_{i \in S} v_i \leq 1 \iff 1 \geq (1-\mu) + \sum_{i \in S} (-v_i) \geq 0.$$

■

We can now specify the sets of directions of increasingness for linear fusion functions.

Proposition 6: Let $L[\mu, \vec{v}]$ be a linear fusion function for some $\mu \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$. Then, the following items hold:

(a) $\mathcal{D}^{\uparrow}(L[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \cdot \vec{v} \geq 0\}$,
(b) $\mathcal{C}(L[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \cdot \vec{v} = 0\}$,
(c) $\mathcal{D}^{\uparrow}_{\text{OD}}(L[\mu, \vec{v}]) = \mathcal{D}^{\uparrow}_{\text{SOD}}(L[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r}_\sigma \cdot \vec{v} \geq 0 \text{ for all } \sigma \in S_n\}$,
(d) $\mathcal{C}_{\text{OD}}(L[\mu, \vec{v}]) = \mathcal{C}_{\text{SOD}}(L[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r}_\sigma \cdot \vec{v} = 0 \text{ for all } \sigma \in S_n\}$.

Proof: Let $\mathbf{x} \in [0,1]^n$, $c > 0$ and $\sigma \in S_n$ such that $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$. Then, since

$$L[\mu, \vec{v}](\mathbf{x} + c\vec{r}) - L[\mu, \vec{v}](\mathbf{x}) = c\vec{r} \cdot \vec{v},$$

we can derive (a) and (b).

For (c) and (d), note that

$$L[\mu, \vec{v}](\mathbf{x} + c\vec{r}_{\sigma^{-1}}) - L[\mu, \vec{v}](\mathbf{x}) = c\vec{r}_{\sigma^{-1}} \cdot \vec{v}.$$

■

Example 3:

(1) Let $f\colon [0,1]^n \to [0,1]$ be the constant function given by $f(\mathbf{x}) = c$, for all $\mathbf{x} \in [0,1]^n$. Then, $f \equiv L[c, \mathbf{0}]$.
(2) Let $\mathbf{w} = (w_1, \dots, w_n) \in [0,1]^n$ be a weight vector, i.e., $\sum_{i=1}^n w_i = 1$, and let $f : [0,1]^n \to [0,1]$ be the weighted average given by $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{w}$ if $\mathbf{x} \in [0,1]^n$. Then, $f \equiv L[0, \mathbf{w}]$.
(3) Let $\mathbf{w} = (w_1, \dots, w_n) \in [0,1]^n$ be a weight vector as in (2) and let $f : [0,1]^n \to [0,1]$ be given by $f(\mathbf{x}) = 1 - \mathbf{x} \cdot \mathbf{w}$ if $\mathbf{x} \in [0,1]^n$. Then, $f \equiv L[1, -\mathbf{w}] \equiv 1 - L[0, \mathbf{w}]$.
(4) Let $f : [0,1]^2 \to [0,1]$ be given by

$$f(x,y) = \frac{1}{2}(1 - x + y)$$

if $(x,y) \in [0,1]^2$. Then, $f \equiv L\left[\frac{1}{2}, \left(-\frac{1}{2}, \frac{1}{2}\right)\right]$.

We also study the sets of directions of increasingness for each form of monotonicity for the following class of functions, the so-called ordered linear fusion functions.

Definition 7:

Let $\mu \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$ be such that $\mu + \mathbf{x}_\sigma \cdot \vec{v} \in [0,1]$ for all $\mathbf{x} \in [0,1]^n$ and $\sigma \in S_n$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$. Let $OL[\mu, \vec{v}] : [0,1]^n \to [0,1]$ be a function given by

$$OL[\mu, \vec{v}]\,(\mathbf{x}) = \mu + \mathbf{x}_\sigma \cdot \vec{v} = \mu + \sum_{i=1}^n x_{\sigma(i)} v_i.$$

We call the function $OL[\mu, \vec{v}]$ an ordered linear fusion function.

Proposition 7: Let $OL[\mu, \vec{v}]$ be an ordered linear fusion function, then $OL[\mu, \vec{v}](\mathbf{x}_\sigma) = OL[\mu, \vec{v}](\mathbf{x})$ for all $\sigma \in S_n$, i.e., it is symmetric.

Let us now present the ordered linear fusion functions analogue result of Proposition 5. The proof of this fact can be found in [20].

Theorem 6: The function $OL[\mu, \vec{v}]$ is an ordered linear fusion function if and only if $0 \leq \mu \leq 1$ and

$$0 \leq \mu + \sum_{i=1}^r v_i \leq 1 \text{ for all } r \in \{1, \dots, n\}.$$

Corollary 2: If $OL[\mu, \vec{v}]$ is an ordered linear fusion function, the function $OL[1-\mu, -\vec{v}]$ is also an ordered linear fusion function.

We also study the conditions under which an ordered linear fusion function and a linear fusion function coincide in the following result.

Proposition 8: Let $L[\eta, \vec{u}]$ be a linear fusion function and $OL[\mu, \vec{v}]$ an ordered linear fusion function. Then, $L[\eta, \vec{u}](\mathbf{x}) = OL[\mu, \vec{v}](\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$ if and only if $\mu = \eta$ and $\vec{u} = \vec{v} = (c, \dots, c) \in \mathbb{R}^n$ where $\mu \in [0,1]$ and $\mu + nc \in [0,1]$.

The next result shows the sets $\mathcal{D}^{\uparrow}_{OD}$ and $\mathcal{C}_{OD}$ for an ordered linear fusion function.

Proposition 9: Let $OL[\mu, \vec{v}] \in \mathbb{R} \times \mathbb{R}^n$ be an ordered linear fusion function. Then, the following hold (1)

1) $\mathcal{D}^{\uparrow}_{\text{OD}}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \cdot \vec{v} \geq 0\}$.
2) $\mathcal{C}_{\text{OD}}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \cdot \vec{v} = 0\}$.

Proof: Let $\mathbf{x} \in [0,1]^n$, $c > 0$ and $\sigma \in S_n$ such that $\mathbf{x}_\sigma$, $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)}$. Thus,

$$F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) = \mu + (\mathbf{x}_\sigma + c\vec{r}) \cdot \vec{v} = F(\mathbf{x}) + c\vec{r} \cdot \vec{v}.$$

■

To characterize the remaining sets of directions for directional and SOD monotonicity, we consider the case $n = 2$, where for all $\sigma \in S_2$ it holds that $\sigma^{-1} = \sigma$. The following result characterizes these sets for $n = 2$. For the proof, see [20].

Theorem 7: Let $OL[\mu, \vec{v}]$ be an ordered linear fusion function for $\mu \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^2$. If we denote $\vec{v} = (v_1, v_2)$ and $\vec{v}^{\,d} = (v_2, v_1)$, we get the following:

(a) $\mathcal{D}^{\uparrow}_{\text{OD}}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} \geq 0\}$;
(b) $\mathcal{C}_{\text{OD}}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} = 0\}$;
(c) $\mathcal{D}^{\uparrow}_{\text{SOD}}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^2_{(\geq)} \mid \vec{r} \cdot \vec{v} \geq 0\} \cup \{\vec{r} \in \mathbb{R}^2_{(\leq)} \mid \vec{r} \cdot \vec{v} \geq 0 \text{ and } \vec{r} \cdot \vec{v}^{\,d} \geq 0\}$;
(d) $\mathcal{C}_{\text{SOD}}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^2_{(\geq)} \mid \vec{r} \cdot \vec{v} = 0\} \cup \{\vec{r} \in \mathbb{R}^2_{(\leq)} \mid \vec{r} \cdot \vec{v} = \vec{r} \cdot \vec{v}^{\,d} = 0\}$;
(e) $\mathcal{D}^{\uparrow}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} \geq 0 \text{ and } \vec{r} \cdot \vec{v}^{\,d} \geq 0\}$;
(f) $\mathcal{C}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} = \vec{r} \cdot \vec{v}^{\,d} = 0\}$.

In the next corollary we show how, in some occasions, the expressions of Theorem 7 can be simplified.

Corollary 3: Let $OL[\mu, \vec{v}]$ be an ordered linear fusion function for $\mu \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^2$.

(a) If $v_1 \geq v_2$ then

$$\mathcal{D}^{\uparrow}_{\text{SOD}}(OL[\mu, \vec{v}]) = \mathcal{D}^{\uparrow}_{\text{OD}}(OL[\mu, \vec{v}]) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} \geq 0\}.$$

(b) If $\vec{r} \cdot \vec{v} = 0$, then (b1) If $v_1 \neq v_2$, then

$$\begin{aligned}\mathcal{C}_{\text{SOD}}(OL[\mu,\vec{v}]) &= \mathcal{C}(OL[\mu,\vec{v}]) \\ &= \{\vec{r} \in \mathbb{R}^2_{(\geq)} \mid \vec{r}\cdot\vec{v} = 0\}\,.\end{aligned}$$

(b2) If $v_1 = v_2$, then

$$\begin{aligned}\mathcal{C}_{\text{SOD}}(OL[\mu,\vec{v}]) &= \mathcal{C}_{\text{OD}}(OL[\mu,\vec{v}]) \\ &= \mathcal{C}(OL[\mu,\vec{v}]) \\ &= \{\vec{r} \in \mathbb{R}^2 \mid \vec{r}\cdot\vec{v} = 0\}\,.\end{aligned}$$

Proof:

(a) If $v_1 \geq v_2$, then if $r_1 \leq r_2$ we have that $\vec{r}\cdot\vec{v}^{\,d} \geq \vec{r}\cdot\vec{v}$.
(b) The result follows the fact that if $\vec{r}\cdot\vec{v} = 0$, then $\vec{r}\cdot\vec{v}^{\,d} = (r_2 - r_1)(v_1 - v_2)$.

■

In what follows we show a collection of examples of ordered linear fusion functions.

Example 4: Let $\lambda \in [0,1]$, then the function

$$OL[0,(1,-\lambda)]\colon [0,1]^2 \to [0,1]$$

is given by

$$OL[0,(1,-\lambda)](\mathbf{x}) = \max(x_1,x_2) - \lambda\min(x_1,x_2)\,.$$

Consequently, the function $OL[1,(-1,\lambda)]$ is given by

$$OL[1,(-1,\lambda)](\mathbf{x}) = 1 - \max(x_1,x_2) + \lambda\min(x_1,x_2)\,.$$

Example 5: The function

$$OL[0,(1,-1)]\colon [0,1]^2 \to [0,1]$$

is given by

$$OL[0,(1,-1)](\mathbf{x}) = |x_1 - x_2|\,.$$

Consequently, the function $OL[1,(-1,1)]$ is given by

$$OL[1,(-1,1)](\mathbf{x}) = 1 - |x_1 - x_2|\,.$$

Observe that the function $OL[1,(-1,1)]$ is a restricted equivalence function [4] that belongs to the family of restricted equivalence function given by

$$REF(x_1,x_2)1 - |x_1 - x_2|^p \text{ for some } p > 0. \tag{2}$$

Note that, for a given $p > 0$, we can set the function $f : [0,1]^2 \to [0,1]$ given by

$$f(x_1,x_2) = |x_1 - x_2|^p$$

for $(x_1,x_2) \in [0,1]^2$. Then, considering the function $\varphi(x) = x^p$ as in Proposition 4 and Theorem 7 we deduce that

- $\mathcal{D}^{\uparrow}_{\text{SOD}}(f) = \mathcal{D}^{\uparrow}_{\text{OD}}(f) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 \geq r_2\}$;
- $\mathcal{D}^{\uparrow}(f) = \mathcal{C}(f) = \mathcal{C}_{\text{SOD}}(f) = \mathcal{C}_{\text{OD}}(f) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 = r_2\}$.

Clearly, if we have a function $REF : [0,1]^2 \to [0,1]$ as in (2), then it holds that $REF(x_1,x_2) = 1 - f(x_1,x_2)$, and it is straightforward to check that REF is $\vec{r}$-increasing (resp. OD $\vec{r}$-increasing, SOD $\vec{r}$-increasing) if and only if f is $\vec{r}$-decreasing (resp. OD $\vec{r}$-decreasing, SOD $\vec{r}$-decreasing) for some $\vec{r} \in \mathbb{R}^n$.

Additionally, on the one hand, one can show that a vector $\vec{s} = (s,s) \in \mathcal{D}^{\uparrow}_{\text{SOD}}(REF)$ and, on the other hand, if we consider $\mathbf{y} = (y,y)$ such that $0 < y < 1$ and assume that REF is SOD $\vec{r}$-increasing, if we take $c > 0$ such that $\mathbf{y} + c\vec{r} \in [0,1]^2$, we conclude that

$$REF(\mathbf{y} + c\vec{r}) = 1 - |c(r_1 - r_2)|^p \geq REF(\mathbf{y}) = 1.$$

Therefore, $r_1 = r_2$.

Hence, we have proved that

$$D^{\uparrow}_{\text{SOD}}(REF) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 = r_2\}.$$

And taking into account, the relation of $\vec{r}$-increasingness (for directional, OD and SOD monotonicity) with $\vec{r}$-decreasingness of f, the following items hold.

- $\mathcal{D}^{\uparrow}(REF) = \mathcal{D}^{\uparrow}_{\text{SOD}}(REF) = \mathcal{C}(REF) = \mathcal{C}_{\text{SOD}}(REF) = \mathcal{C}_{\text{OD}}(REF) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 = r_2\}$;
- $\mathcal{D}^{\uparrow}_{\text{OD}}(REF) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 \leq r_2\}$.

The following are some other examples of $OL[\mu,\vec{v}]$ functions.

Example 6: The function

$$OL\left[\frac{1}{2},(\frac{1}{2},-1)\right] : [0,1]^2 \to [0,1]$$

is given by

$$OL\left[\frac{1}{2},(\frac{1}{2},-1)\right](\mathbf{x}) = \frac{1}{2}(1+\max(x_1-x_2)-2\min(x_1-x_2))\,.$$

Example 7: The function

$$OL\left[1,(-1,\frac{1}{2})\right] : [0,1]^2 \to [0,1]$$

is given by

$$OL\left[1,(-1,\frac{1}{2})\right](\mathbf{x}) = 1 - \max(x_1,x_2) + \frac{1}{2}\min(x_1,x_2)\,.$$

We include the ordered weighted average operator, defined by Yager [23].

Example 8: Let $\mathbf{w} \in [0,1]^n$ with $\sum_{i=1}^n w_i = 1$, the ordered weighted average operator $OWA_{\mathbf{w}}\colon [0,1]^n \to [0,1]$ is given by

$$OWA_{\mathbf{w}}(\mathbf{x}) = \mathbf{x}_\sigma \cdot \mathbf{w},$$

where $\sigma \in S_n$ is such that $\mathbf{x} \in [0,1]^n_{(\geq)}$. Then, it holds that $OWA_{\mathbf{w}} = OL[0,\mathbf{w}]$.

VI. Conclusions

We have studied the most recently introduced forms of monotonicity in the setting of aggregation functions, that are weaker than standard component-wise monotonicity. In particular, we have dealt with weak monotonicity, directional monotonicity, ordered directional monotonicity and strengthened ordered directional monotonicity. We have discussed some of the relevant properties of functions that satisfy these monotonicity conditions and have studied the relation that there exists between each form of monotonicity. Moreover, we have presented two families of functions, the family of linear fusion functions and the family of ordered linear fusion functions, and we have characterized the directions for which the functions in these two families increase (in the directional, ordered directional and strengthened ordered directional sense). We have shown that the very used in decision making problems OWA operators are a particular case of ordered linear fusion functions.

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